# Multicriterial Fractional Optimization

Klaus Tammer *
Humboldt-Universität Berlin, FB Mathematik,
Unter den Linden 6, D-10099 Berlin, Germany
Christiane Tammer and Evelin Ohlendorf
Martin-Luther-Universität Halle-Wittenberg,
FB Mathematik u.Informatik, D-06099 Halle, Germany

**Abstract**

At first we introduce different solution concepts for general vector optimization problems and summarize some relations between them. Further, we apply these solution concepts to vectorial fractional optimization problems and show that the well-known Dinkelbach-transformation can be generalized in the sense, that even in vector optimization exact as well as approximate solutions for the original problem and for the transformed one are closely related. Finally, we discuss possibilities to handle the transformed vector optimization problem by means of parametric optimization.

Keywords and phrases: Solutions and approximate solutions in vector optimization, fractional vector optimization, Dinkelbach-transformation, dialogue approach via parametric optimization

## 1 Introduction

Many aims in real decision problems can be expressed by a fractional objective function (cf. [20]) so that the field of fractional optimization is very actual. For the case of optimization problems with only one fractional objective function Dinkelbach [4] has proposed a parametric solution approach. This approach is based on the relation to a special parametric

*Supported by the Deutsche Forschungsgemeinschaft under grant Gu 304/1-4.

problem, which is described without the original ratios. However, it requires, additionally, the generation of that unknown parameter value, for which equivalence to our original problem holds. Many other authors have already published results to generalize Dinkelbach's idea also for efficient and properly efficient solutions of vectorial optimization problems with m fractional objective functions (cf. Bector and Chandra [1], Kaul and Lyall [12] and Weir [28]). But some of those results are not intirely correct. Moreover, up to now for $m > 1$ a general convergent algorithm to generate the required parameter vector as in the case m=1 does not exist.
The aim of our paper is to extend the results of Dinkelbach and other authors to different sets of approximate efficient and properly efficient solutions of multicriterial fractional optimization problems, which were introduced by Chr. Tammer [23] as well as Dentcheva and Helbig [3]. As a by-product we obtain the corrected formulations of corresponding results for the exact solutions.
In the second section we give a brief survey about different concepts for solutions and approximate solutions in vector optimization. The third section is devoted to the mentioned relations between the (approximate) solutions of the original multicriterial fractional problem and the transformed one. In the forth section we discuss possibilities to solve the transformed problem by a three-level dialogue approach following ideas of the book [9].

## 2 Concepts of Solutions and Approximate Solutions in Vector Optimization

In our paper we assume that $\mathbf{F}$ and $\mathbf{P}$ are nonempty subsets of the m-dimensional Euclidean space $\mathbb{R}^m$. As usual we define the set $Eff(\mathbf{F},\mathbf{P})$ of efficient elements of the set $\mathbf{F}$ with respect to the set $\mathbf{P}$ in the form

$$Eff(\mathbf{F},\mathbf{P}) = \{\bar{\mathbf{y}} \in \mathbf{F} \ / \ \mathbf{F} \cap (\bar{\mathbf{y}} - (\mathbf{P} \setminus \{\mathbf{0}\})) = \emptyset\}.$$

The general vector optimization problem $\mathcal{VOP}$ associated with $\mathbf{F}$ and $\mathbf{P}$ is defined as the task to determine the set $Eff(\mathbf{F},\mathbf{P})$.

As the mostly considered special case of $\mathcal{VOP}$ we mention the situation that

$$\mathbf{F} = f(\mathbf{X}) = \{(f_1(x), ..., f_m(x))' / \ x \in \mathbf{X}\}, \ \mathbf{X} \subseteq \mathbb{R}^n, \mathbf{P} = \mathbb{R}^m_+, \tag{1}$$

in which we write the vector optimization problem $\mathcal{VOP}$ in the form

$$(\mathcal{P}) \qquad f(x) \ \rightarrow \ v - min! \text{ subject to } x \in \mathbf{X}. \tag{2}$$

In the case $\mathbf{P} = I\!R^m_+$ let us write shortly $Eff(\mathbf{F})$ instead of $Eff(\mathbf{F}, I\!R^m_+)$. Beside the set $Eff(\mathbf{F})$ we also want to consider the set $p - Eff(\mathbf{F}, \mathcal{Z})$ of properly efficient elements of $\mathbf{F}$ in the sense of [7] with respect to a family $\mathcal{Z}$ of sets $\mathbf{Q} \subseteq I\!R^m$ satisfying

$$cl\ \mathbf{Q} + (I\!R^m_+ \setminus \{0\}) \subseteq int\ \mathbf{Q} \ \wedge \ 0 \in cl\ \mathbf{Q} \tag{3}$$

defined in the form

$$p - Eff(\mathbf{F}, \mathcal{Z}) = \bigcup_{\mathbf{Q} \in \mathcal{Z}} Eff(\mathbf{F}, \mathbf{Q})$$

as well as the set

$$\begin{array}{rl} G - Eff(\mathbf{F}) = & \{\bar{y} \in \mathbf{F} / \exists c > 0 : \forall y \in \mathbf{F},\ \forall i = 1, ..., m \text{ with } y_i < \bar{y}_i \\ & \exists\ j \in \{1, ..., m\} \setminus \{i\} \text{ with } c(y_j - \bar{y}_j) \geq \bar{y}_i - y_i\} \end{array}$$

of properly efficient elements of $\mathbf{F}$ in the sense of Geoffrion [6].
Some relations between the given concepts are summarized in the following proposition.

**Proposition 2.1** *1. We have already* $p - Eff(\mathbf{F}, \mathcal{Z}) \subseteq Eff(\mathbf{F})$.

2. *If* $\mathbf{F} \subseteq I\!R^m$ *is convex or if* $\mathbf{F}$ *is defined according to (1) with* $\mathbf{X}$ *convex and* $f_i, i = 1,,,.m$ *convex on* $\mathbf{X}$*, then*
$G - Eff(\mathbf{F}) = \{\bar{y} \in \mathbf{F} \ / \ \exists \mu \in int I\!R^m_+ \ \text{ with } \mu'\bar{y} = \min_{y \in \mathbf{F}} \mu' y\}$.

3. *For the case* $\mathbf{P} = I\!R^m_+$, $\mathcal{Z} = \mathcal{Z}_0$ *with* $\mathcal{Z}_0 = \{\mathbf{Q}^l, l = 1, 2, ...\}$,
$\mathbf{Q}^l = \bigcup_{i=1,\dots,m} \mathbf{Q}^l_i$, $\mathbf{Q}^l_i = \{y \in I\!R^m / y_i > 0,\ y_i + \alpha^l y_j > 0, \forall j \neq i\}$,
*where* $\{\alpha^l\}$ *is an arbitrary positive sequence with* $\alpha^l \to \infty$*, we have*
$G - Eff(\mathbf{F}) = p - Eff(\mathbf{F}, \mathcal{Z}_0)$.

**Proof:** Statement 1. follows from $I\!R^m_+ \setminus \{0\} \subseteq int\ \mathbf{Q}$, which is a consequence of (3). The first part of Statement 2. was given in [2], the second one in [6].
To prove Statement 3. let us assume $\bar{y} \in \mathbf{F}$. The condition $\bar{y} \notin G - Eff(\mathbf{F})$ means that for each $c > 0$ there is a point $y(c) \in \mathbf{F}$ and at least one index $i \in \{1, ..., m\}$ with $y_i(c) < \bar{y}_i$ such that for all $j \neq i$ we have $c(y_j(c) - \bar{y}_j) < \bar{y}_i - y_i(c)$. Now let $\{\alpha^l\}$ be an arbitrary positive and divergent sequence. Choosing $c = \alpha^l, l = 1, 2, ...$ we see that $y^l = y(\alpha^l)$ satisfies $\bar{y} - y^l \in \mathbf{Q}^l$ and, hence, $\bar{y} \notin p - Eff(\mathbf{F}, \mathcal{Z}_0)$.
If, on the other hand, $\bar{y} \notin p - Eff(\mathbf{F}, \mathcal{Z}_0)$ then, because of the divergence of the sequence $\{\alpha^l\}$, for every $c > 0$, there is a number l with $\alpha^l \geq c$ such

that $\bar{y}-y^l \in \mathbf{Q}^l$ implies $c(y^l_j-\bar{y}_j) < \bar{y}_i - y^l_i$ for a certain index i with $y^l_i < \bar{y}_i$ and all $j \neq i$ and, hence, $\bar{y} \notin G - Eff(\mathbf{F})$. q.e.d.

Note that the set $p - Eff(\mathbf{F}, \mathcal{Z}_0)$ in Statement 3. does not depend on the concrete choice of the sequence $\{\alpha^l\}$.
A very similar result as Statement 3. was given in chapter 5.2 in [27].

Different concepts for approximate solutions of $\mathcal{VOP}$ were introduced in [3], [5], [8], [16], [17], [18], [22], [23], [26]. We follow here the papers [3] and [23] and introduce three different types of approximate solutions for the special case that $\mathbf{P} = \mathbb{R}^m_+$, where $k \in int\mathbb{R}^m_+$ and $\epsilon \geq 0$.

According to [23] the set of $\epsilon$-efficient elements of $\mathbf{F}$ with respect to k is given by

$$\epsilon - Eff(\mathbf{F}, k) = Eff(\mathbf{F}, \mathbb{R}^m_+ + \epsilon k)$$

and the set of properly $\epsilon$-efficient elements of $\mathbf{F}$ with respect to $\mathcal{Z}$ and k is given by

$$\epsilon p - Eff(\mathbf{F}, \mathcal{Z}, k) = \bigcup_{\mathbf{Q}\in\mathcal{Z}} Eff(\mathbf{F}, \mathbf{Q} + \epsilon k),$$

where $\mathcal{Z}$ is defined as in (3).

Under the assumption that $z : \mathbb{R}^m \to \mathbb{R}$ is monotone with respect to $\mathbb{R}^m_+$ $(y^1 \leq y^2 \Rightarrow z(y^1) \leq z(y^2))$ the set of $\epsilon$-efficient elements of $\mathbf{F}$ with respect to z according to [3] is defined by

$$\epsilon - Eff(\mathbf{F}, z) = \{\bar{y} \in \mathbf{F} \;/\; y \leq \bar{y} \Longrightarrow z(\bar{y}) \leq z(y) + \epsilon\}.$$

**Proposition 2.2** *For any $k \in int\mathbb{R}^m_+$ and any $\epsilon \geq 0$ we have:*

1. $Eff(\mathbf{F}) \subseteq \epsilon - Eff(\mathbf{F}, k)$.
2. $p - Eff(\mathbf{F}, \mathcal{Z}) \subseteq \epsilon p - Eff(\mathbf{F}, \mathcal{Z}, k)$.
3. $\epsilon p - Eff(\mathbf{F}, \mathcal{Z}, k) \subseteq \epsilon - Eff(\mathbf{F}, k)$.
4. *For any $\bar{y} \in \epsilon - Eff(\mathbf{F}, k)$ the functional $\hat{z}$ given by $\hat{z}(y) = z_0(y - \bar{y})$, where*

$$z_0(y) = \inf\{t \in \mathbb{R} \;/\; y \in -clB + tk\} \tag{4}$$

*for $B = \mathbb{R}^m_+$ is strictly monotone with respect to $int\mathbb{R}^m_+$ and $\bar{y} \in \epsilon - Eff(\mathbf{F}, \hat{z})$.*

5. *Let the functional $z_0$ defined in (4) with $cl B + int \mathbb{R}^m_+ \subseteq int B$ be strictly monotone with respect to $\mathbb{R}^m_+$, subadditive and continuous. Then there exists an open set $\mathbf{Q} \subseteq \mathbb{R}^m$ with $\mathbb{R}^m_+ \setminus \{0\} \subseteq \mathbf{Q}$, $0 \in cl\ \mathbf{Q} \setminus \mathbf{Q}$, $cl\ \mathbf{Q} + (\mathbb{R}^m_+ \setminus \{0\}) \subseteq \mathbf{Q}$ such that $\epsilon - Eff(\mathbf{F}, z_0) \subseteq \epsilon - Eff(\mathbf{F}, \mathbf{Q}, k)$.*

**Proof:** Statements 1. and 2. follow from the fact that according to our assumptions $\mathbb{R}^m_+ + \epsilon k \subseteq int \mathbb{R}^m_+$ holds. In the same way as Statement 1. from Proposition 2.1 also Statement 3. of Proposition 2.2 is a consequence of $\mathbb{R}^m_+ \setminus \{0\} \subseteq int \mathbf{Q}$, which implies $\mathbb{R}^m_+ + \epsilon k \setminus \{0\} \subseteq int(\mathbf{Q} + \epsilon k)$. Statements 4. and 5. were proved in [24]. q.e.d.

# 3 Generalized Dinkelbach-Transformation

Consider as a special case of $(\mathcal{P})$ a vectorial fractional optimization problem

$$(\mathcal{P}_f) \qquad f(x) = \frac{g(x)}{h(x)} \ \rightarrow \ v - min! \qquad \text{subject to } x \in \mathbf{X} \subseteq \mathbb{R}^n,$$

where $\frac{g(x)}{h(x)} := (\frac{g_1(x)}{h_1(x)}, ..., \frac{g_m(x)}{h_m(x)})'$ and $h_i(x) > 0 \ \forall \ x \in \mathbf{X}, \ i = 1, ..., m$.
We show, that $(\mathcal{P}_f)$ is closely related to a multiparametric vector optimization problem $\mathcal{P}(\lambda)$, which we call the corresponding Dinkelbach-transformed problem, namely

$$\mathcal{P}(\lambda): \qquad H(x, \lambda) \ \rightarrow v - min! \qquad \text{subject to } x \in \mathbf{X},$$

where $H_i(x, \lambda) = g_i(x) - \lambda_i h_i(x), \ i = 1, ..., m$ and $\lambda \in \mathbb{R}^m$ is a parameter which must be choosen in a suitable way.

The original result of Dinkelbach [4] from 1967 (and also the foregoing result of Jagannathan [10] from 1966 for linear fractional problems) concerns the case m=1 with only one objective function and says that a given point $\bar{x}$ is optimal for $(\mathcal{P}_f)$ iff it is optimal for $\mathcal{P}(\bar{\lambda})$ with $\bar{\lambda} = \frac{g(\bar{x})}{h(\bar{x})}$.
Corresponding results for the sets of efficient and properly efficient solutions, respectively, of both problems in the case $m \geq 1$ were given by Bector and Chandra [1], Kaul and Lyall [12], Weir [28] and others. Note that the formulation as well as the proof of the corresponding Lemma 1 in [12] and Theorem 4 in [27] are not entirely correct in the given form. Above all, the authors disregarded the fact that in the case of proper efficiency it is essential to assume, additionally, that all ratios $\frac{h_i}{h_j}$ are bounded below by positive bounds (and not only by zero).

Now, in the following two theorems we formulate the relations between the sets of approximate solutions of $(\mathcal{P}_f)$ and $\mathcal{P}(\lambda)$. Proposition 2.2 gives a possibility to extend the results also to the set of approximate solutions in the sense of [3].

**Theorem 3.1** *Let $k \in int\mathbb{R}^m_+, \epsilon \geq 0$ and $\bar{x} \in \mathbf{X}$. Then we have*
$f(\bar{x}) \in \epsilon - Eff(f(\mathbf{X}), k) \iff H(\bar{x}, \bar{\lambda}) \in \epsilon - Eff(H(\mathbf{X}, \bar{\lambda}), \bar{k})$ *for*

$$\bar{\lambda}_i = \frac{g_i(\bar{x})}{h_i(\bar{x})} - \epsilon k_i \qquad and \qquad \bar{k}_i = k_i h_i(\bar{x}), \qquad \forall i = 1, ..., m. \tag{5}$$

**Proof:** The relation $H(\bar{x}, \bar{\lambda}) \notin \epsilon - Eff(H(\mathbf{X}, \bar{\lambda}), \bar{k})$ is equivalent with the existence of an element $x^1 \in \mathbf{X}$ with

$$g_i(x^1) - \bar{\lambda}_i h_i(x^1) \leq g_i(\bar{x}) - \bar{\lambda}_i h_i(\bar{x}) - \epsilon \bar{k}_i \qquad \forall i = 1, ..., m,$$

where for at least one index i the corresponding inequality must be strict. Dividing these inequalities by $h_i(x^1)$ and taking relation (5) into account we get the equivalent inequalities

$$\frac{g_i(x^1)}{h_i(x^1)} \leq \frac{g_i(\bar{x})}{h_i(\bar{x})} - \epsilon k_i \qquad \forall i = 1, ..., m,$$

where again for at least one index i the corresponding inequality must be strict. But this is equivalent to $f(\bar{x}) \notin \epsilon - Eff(f(\mathbf{X}), k)$. q.e.d.

For the special case $\epsilon = 0$ we get the already mentioned result of [1] and [12] in the corrected form (namely, including the essential condition (5) for $\epsilon = 0$, which actually was used there in the proofs but has been forgotten in the formulation of the statement).

**Corollary 3.1** $f(\bar{x}) \in Eff(f(\mathbf{X})) \iff H(\bar{x}, \bar{\lambda}) \in Eff(H(\mathbf{X}, \bar{\lambda}))$ *for* $\bar{\lambda}$ *according to (5) with $\epsilon = 0$.*

**Theorem 3.2** *Let be $k \in int\mathbb{R}^m_+, \epsilon \geq 0$ and $\bar{x} \in \mathbf{X}$ and assume that there is a positive number $\gamma$ such that for all i,j= 1,..., m and all $x \in \mathbf{X}$ it holds $\frac{h_i(x)}{h_j(x)} \geq \gamma$. Then we have*
$f(\bar{x}) \in \epsilon p - Eff(f(\mathbf{X}), \mathcal{Z}_0, k) \Leftrightarrow H(\bar{x}, \bar{\lambda}) \in \epsilon p - Eff(H(\mathbf{X}, \bar{\lambda}), \mathcal{Z}_0, \bar{k})$ *for* $\bar{\lambda}$ *and* $\bar{k}$ *according to (5).*

**Proof:** The relation $H(\bar{x}, \bar{\lambda}) \notin \epsilon p - Eff(H(\mathbf{X}, \bar{\lambda}), \mathcal{Z}_0, \bar{k})$ implies that for any l=1,2,... there is a point $x^l \in \mathbf{X}$ with $H(\bar{x}, \bar{\lambda}) - H(x^l, \bar{\lambda}) - \epsilon \bar{k} \in \mathbf{Q}^l$.

This means that there is an index i with $(H_i(\bar{x},\bar{\lambda}) - H_i(x^l,\bar{\lambda}) - \epsilon\bar{k}_i) > 0$ such that $\forall j \neq i$ and for a certain positive and divergent sequence $\{\alpha^l\}$ we have

$$(H_i(\bar{x},\bar{\lambda}) - H_i(x^l,\bar{\lambda}) - \epsilon\bar{k}_i) \ + \ \alpha^l(H_j(\bar{x},\bar{\lambda}) - H_j(x^l,\bar{\lambda}) - \epsilon\bar{k}_j) \ > 0.$$

Taking into account the definition of H and of $\bar{\lambda}$ and $\bar{k}$ in (5) these inequalities can be written in the form

$$(\frac{g_i(\bar{x})}{h_i(\bar{x})}h_i(x^l) - g_i(x^l) - \epsilon\bar{k}_i) \ + \ \alpha^l(\frac{g_j(\bar{x})}{h_j(\bar{x})}h_j(x^l) - g_j(x^l) - \epsilon\bar{k}_j) > 0,$$

or, equivalently, in the form

$$(f_i(\bar{x}) - f_i(x^l) - \epsilon k_i) \ + \ \frac{h_j(x^l)}{h_i(x^l)}\alpha^l(f_j(\bar{x}) - f_j(x^l) - \epsilon k_j) \ > 0.$$

Together with $(f_i(\bar{x}) - f_i(x^l) - \epsilon k_i) > 0$ , this implies

$$f_i(\bar{x}) - f_i(x^l) - \epsilon k_i \ + \ \beta^l(f_j(\bar{x}) - f_j(x^l) - \epsilon k_j) > 0,$$

where $\beta^l = \gamma\alpha^l \to \infty$. But this means $f(\bar{x}) \notin \epsilon p - Eff(f(\mathbf{X}), \mathcal{Z}_0, k)$ .

On the same way the assumption $f(\bar{x}) \notin \epsilon p - Eff(f(\mathbf{X}), \mathcal{Z}_0, k)$ implies the existence of points $x^l \in \mathbf{X}$ for l=1,2,... satisfying for at least one index i the relation $(H_i(\bar{x},\bar{\lambda}) - H_i(x^l,\bar{\lambda}) - \epsilon\bar{k}_i) > 0$ and for all $j \neq i$

$$(H_i(\bar{x},\bar{\lambda}) - H_i(x^l,\bar{\lambda}) - \epsilon\bar{k}_i) \ + \ \beta^l(H_j(\bar{x},\bar{\lambda}) - H_j(x^l,\bar{\lambda}) - \epsilon\bar{k}_j) \ > 0,$$

where $\beta^l = \gamma\alpha^l \to \infty$ such that $H(\bar{x},\bar{\lambda}) \notin \epsilon p - Eff(H(\mathbf{X},\bar{\lambda}), \mathcal{Z}_0, \bar{k})$. q.e.d.

Note that the assertion of Theorem 3.2 does not remain true if we only assume (as it was done in [12] and [28]) $h_i(x) > 0$ on $\mathbf{X}$ for i=1,...,m, since then $inf\{\frac{h_i(x)}{h_j(x)}/x \in \mathbf{X}\} = 0$ is not excluded.
This can be seen by the following small example. Let be $n = 1, m = 2, \mathbf{X} = \{x \in I\!R/x \geq 0\}, g_1(x) = e^{-x},\ h_1 = 1,\ g_2(x) = x^2 + x + 1,\ h_2(x) = x^2 + 1$. Then, for instance, $\bar{x} = 0$ yields a properly efficient element (in the sense of Geoffrion) $H(\bar{x},\bar{\lambda}) = (0,0)'$ for $\mathcal{P}(\bar{\lambda})$ with $\bar{\lambda} = (1,1)'$ but $(\frac{g_1(\bar{x})}{h_1(\bar{x})}, \frac{g_2(\bar{x})}{h_2(\bar{x})})' = (1,1)'$ is not properly efficient in the sense of Geoffrion for $(\mathcal{P}_f)$. Similar examples can also be constructed for the other direction of Theorem 3.2.
Of course, if all functions $h_i$ are equal or if there are positive lower and upper bounds for all functions $h_i$ on $\mathbf{X}$, the required boundedness of all ratios $\frac{h_i}{h_j}$ by positive bounds is satisfied.

For the special case $\epsilon = 0$ we get the corrected formulation of the inexact results in [12] and [28].

**Corollary 3.2** *If there is a positive number $\gamma$ such that for all i,j=1,...,m and all $x \in \mathbf{X}$ it holds $\frac{h_i(x)}{h_j(x)} \geq \gamma$ then we have*
$f(\bar{x}) \in G-Eff(f(\mathbf{X})) \iff H(\bar{x}, \bar{\lambda}) \in G-Eff(H(\mathbf{X}, \bar{\lambda}))$ *for* $\bar{\lambda}$ *according to (5).*

# 4 Possibilties for a Solution Approach

The reason of using models of vector optimization for solving concrete decision problems is the fact that very often it is impossible to formulate the interests of the decision maker a priori by only one objective function. As a natural consequence of such an incomplete knowledge about the underlying decision problem we can observe the phenomenon that in vector optimization we get a great number of "solutions", enjoying a priori the same rights. Of course, in practical decision problems the final aim must be to find such a feasible decision which corresponds to the decision maker's interests in a certain "optimal" way.
As already described in [9] this can often be realized by organizing a learning process in form of a dialogue procedure in which one can compute and compare as much solutions as nessesary to help the decision maker to express his individual interests more precisely. Such a dialogue procedure is usually a certain kind of a two-level algorithm and needs essentially a suitable parametric surrogate optimization problem related to the underlying vector optimization problem.
Theoretically, all these ideas can also be applied directly to the fractional vector optimization problem $(\mathcal{P}_f)$ studied in the foregoing section. However, there may be computational difficulties to handle problems with complicated fractional objective functions. Moreover, there are also theoretical difficulties to ensure convexity properties of the surrogate problem which to be solved in such a dialogue procedure. Note that even linear fractionals are not convex but only pseudoconvex and that for sums of fractionals even generalized convexity properties do not hold anymore. Hence, for instance, Statement 2. of Proposition 2.1 cannot be used.
For this reason we want to discuss here possibilities to apply a dialogue procedure not directly to the original fractional problem $(\mathcal{P}_f)$ but to the corresponding Dinkelbach-transformed problem $\mathcal{P}(\lambda)$. However, in such an approach we have to overcome another difficulty, namely the generation of the essential parameter value $\bar{\lambda}$ satisfying (5). Hence, different to dialogue procedures in the usual case, for our considered case of fractional vector optimization problems we propose a three-level dialogue procedure.
Let us explain our ideas for the mostly used set $Eff(\mathbf{F})$ of efficient solu-

tions ( $\epsilon = 0$ ) of $(\mathcal{P}_f)$ and the mostly used surrogate problem, in which the artificial objective function is the weighted sum of the original objective functions. Applied to $\mathcal{P}(\lambda)$ our parametric surrogate problem has the form

$$\mathcal{P}(\lambda, \mu): \qquad F(x, \lambda, \mu) \ \rightarrow \ min! \qquad \text{subject to } \ x \in \mathbf{X},$$

where $\mu > 0$ and

$$F(x, \lambda, \mu) = \sum_{i=1}^{m} \mu_i (g_i(x) - \lambda_i h_i(x)).$$

The already mentioned three levels of a dialogue procedure for $(\mathcal{P}_f)$ may be characterized in the following way.

Level 1: Compare all stored results and decide to stop the procedure or not. If not, choose a new parameter vector $\mu$.

Level 2: Find for the value of $\mu$ given from Level 1 a vector $\lambda$ such that there exists a solution x of $\mathcal{P}(\lambda, \mu)$ satisfying $H(x, \lambda) = 0$.

Level 3: Find for the values $\mu$ and $\lambda$ given in the Levels 1 and 2 a solution x of $\mathcal{P}(\lambda, \mu)$ satisfying $H(x, \lambda) = 0$ and store x together with additional informations on x (especially the vector $\frac{g(x)}{h(x)}$). Go to Level 1.

Level 1 is the pure dialogue part in which we have to generate a new parameter value $\mu$ as long as we are not satisfied with the generated results. Level 3 can often be realized successfully by pathfollowing methods of parametric optimization. Because of the fact that possibilities to realize the levels 1 and 3 are already described extensively in former papers (cf. [9]) we concentrate our considerations here on the second level. The typical difficulty in this level is the fact that the essential equation $(5)_0$ is only given implicitely since the solution x of the third level is unknown at the time in which we have to solve the second level.

Let us study Level 2 under the following additional assumption A4). Here we take the symbol $\mathbf{PC}^r$, $(r \geq 1)$ to denote the class of those $\mathbf{C}^{r-1}$-functions for which the derivations of order r-1 are piecewise $\mathbf{C}^1$ (cf. [21], [25]). Moreover, we use the concept of strong stability of stationary points described by Kummer ([15], Section 5) which generalizes corresponding concepts already known for problems described by $\mathbf{C}^2$-functions (cf. Kojima [13]).

A4) $\mathbf{X} = \{x \in I\!R^n \ / \ q_l(x) \leq 0, \ l = 1, ..., p\}$, $g_i, h_i, q_l \in \mathbf{PC}^2$ and there exists a strongly stable stationary point $(x^*, u^*)$ of $\mathcal{P}(\lambda^*, \mu^*)$ satisfying condition $(5)_0$.

**Proposition 4.1** *Let us assume A4). Then we have:*

1. *There are neighbourhoods* $\mathbf{U}$ *of* $\lambda^*, \mathbf{V}$ *of* $\mu^*$ *and* $\mathbf{W}$ *of* $(x^*, u^*)$ *such that for each* $(\lambda, \mu) \in \mathbf{U} \times \mathbf{V}$ *problem* $\mathcal{P}(\lambda, \mu)$ *has a unique stationary point* $(\bar{x}(\lambda, \mu), \bar{u}(\lambda, \mu))$ *of in* $\mathbf{W}$.
2. *The vector function* $(\bar{x}, \bar{u})$ *belongs to the class* $\mathbf{PC}^1$ *on* $\mathbf{U}$.
3. *The vector function* $G$*, defined on* $\mathbf{U}$ *by*

$$G_i(\lambda) = g_i(\bar{x}(\lambda, \mu^*)) - \lambda_i h_i(\bar{x}(\lambda, \mu^*)), \ \ i = 1, ..., m$$

   *belongs to the class* $\mathbf{PC}^1$.

**Proof:** Statements 1. and 2. follow from an implicite function theorem of [15]. Statement 3. is a consequence of Statement 2. and a chain rule given in [21] and [25]. q.e.d.

Obviously, under A4) condition $(5)_0$ can be reformulated in the form

$$G(\lambda) = 0, \qquad \lambda \in \mathbf{U}, \tag{6}$$

To solve (6) we can apply suitable generalizations of the Newton-method for nonsmooth equations using generalized derivatives. In the papers [21] and [25] one can find possibilities to generate the generalized Jacobian of the vector functions $\bar{x}$ and G. To ensure convergence to the (of course unknown) point $\lambda^*$ from assumption A4) usually one needs a suitable initial point $\lambda^0$ in a sufficiently small neighbourhood of $\lambda^*$.
Among the great number of contributions concerning generalizations of the Newton-method to nonsmoth equations we refer here to the rather general results given in [14] and [19]. Usefull ideas to guarantee convergence in the second level even in the case that only approximate solutions of the third level may be generated (what may often be the case) can be found in [25].

Applying Lemma 2.1 in [11] the function $\bar{\varphi}(\lambda, \mu) = F(\bar{x}(\lambda, \mu), \lambda, \mu)$ belongs to the class $\mathbf{PC}^2$ and for each $(\lambda, \mu) \in \mathbf{U} \times \mathbf{V}$ it holds $\frac{\partial}{\partial \lambda_i} \bar{\varphi}(\lambda, \mu) = -\mu_i h_i(\bar{x}(\lambda, \mu))$ and, hence,

$$\sum_{j=1}^{m} \mu_j^* \frac{\partial}{\partial \lambda_i} G_j(\lambda) = -\mu_i^* h_i(\bar{x}(\lambda)) \tag{7}$$

with $\bar{x}(\lambda) = \bar{x}(\lambda, \mu^*)$. Hence, for the special case m=1 (in which we can put without loss of generality $\mu = 1$) the function G belongs even to the class $\mathbf{PC}^2$ with $\nabla G(\lambda) = -h(\bar{x}(\lambda))$. On this way the iteration rule of the Newton-method has the very simple form

$$\lambda^{s+1} = \lambda^s - \frac{g(\bar{x}(\lambda^s)) - \lambda^s h(\bar{x}(\lambda^s))}{-h(\bar{x}(\lambda^s))} = f(\bar{x}(\lambda^s)), \tag{8}$$

which is nothing else than the iteration rule of Dinkelbach [4], who used this rule also under other assumptions as given in A4), since convergence results are very much easier to obtain in the one-dimensional case.
Unfortunately, for $m > 1$ a formula of the type $\nabla G(\lambda) = -h(\bar{x}(\lambda))$ does not follow from (7).

Acknowledgement: The authors thank the referee for many valuable hints.

# References


[1] C.R. Bector, S. Chandra: Multiobjective fractional programming duality: a parametric approach. Research Report, Faculty of Management, University of Manitoba, 1987.

[2] J. Borwein: Proper efficient points for maximizations with respect to cones. SIAM J. Control Opt.15, 1977, 57-63.

[3] D. Dentcheva, S. Helbig, S.: On several concepts for ffl-efficiency. OR Spektrum 16, 1994, 179-186.

[4] W. Dinkelbach: On nonlinear fractional programming. Man. Sci. 13, 7, 1967, 492-498.

[5] J. Ester: Systemanalyse und mehrkriterielle Entscheidung. Verlag Technik, Berlin, 1987.

[6] A. M. Geoffrion: Proper efficiency and the theory of vector maximization. J.Math.Anal.Appl. 22, 1968, 618-630.

[7] Chr. Gerstewitz (Tammer), E. Iwanow: Dualität fr nichtkonvexe Vektoroptimierungsprobleme. Wiss.Z. TH Ilmenau 31, 2, 1985, 61-81.

[8] Chr. Gerth (Tammer): Näherungslösungen in der Vektoroptimierung. Seminarbericht der Sektion Mathematik d. Humboldt-Universität zu Berlin 90, 1987, 67-76.

[9] J. Guddat, F. Guerra Vasquez, K. Tammer, K. Wendler: Multiobjective and Stochastic Optimization Based on Parametric Optimization. Akademie-Verlag, Berlin, 1985.

[10] R. Jagannathan: On some properties of programming problems in parametric form pertaining to fractional programming. Management. Sci. 12, 1966, 609-615.

[11] H. Th. Jongen, Th. Möbert, K. Tammer: On iterated minimization in nonconvex optimization. Math. Oper. Res. 11, 1986, 679-691.

[12] R. N. Kaul, V. Lyall: A note on nonlinear fractional vector maximization. Opsearch 26, 2, 1989, 108-121.

[13] M. Kojima: Strongly stable stationary solutions in nonlinear programs. In: "Analysis and Computation of Fixed Points" (S. M. Robinson, ed.), Academic Press, New York, 1980, 93-138.

[14] B. Kummer: Newton's method based on generalized derivatives for nonsmooth functions: Convergence analysis. In: "Advances in Optimization", Proceedings, Lambrecht, 1991 (W. Oettli and D. Pallaschke, eds.), Springer-Verlag, Berlin, 1992, 171-194.

[15] B. Kummer: Lipschitzian inverse functions, directional derivatives and application in $C^{1,1}$-optimization. JOTA 70, 3, 1991, 559-580.

[16] P. Loridan: ffl-solutions in vector minimization problems. JOTA 43, 2, 1984, 265-276.

[17] A.B. Nemeth: Between pareto efficiency and pareto ffl -efficiency. Optimization 20, 5, 1989, 615-637.

[18] V.V. Podhinovski, V.D. Nogin: Pareto-optimal solutions of multicriterial problems (in russ.). Nauka, Moscow, 1982.

[19] D. Ralph, S. Scholtes: Sensitivity analysis and Newton's method for composite piecewise smooth equations. manuscript, 1995.

[20] S. Schaible, T. Ibaraki: Fractional programming. Europ. Journ. Oper. Res. 12, 1983, 325-338.

[21] S. Scholtes Introduction to piecewise differentiable equations. Preprint No. 53/1994, Universität Karlsruhe, Institut f. Statistik u. Math. Wirtschaftstheorie, 1994.

[22] T. Staib: On two generalizations of Pareto-minimality. JOTA 59, 1988, 289-306.

[23] Chr. Tammer: A generalization of Ekeland's variational principle. Optimization 25, 1992, 129-141.

[24] Chr. Tammer: Stability results for approximately efficient solutions. OR Spektrum 16, 1994, 47-52.

[25] K. Tammer: Two-level optimization with approximate solutions in the lower level. ZOR 41, 1995, 231-249.

[26] I. Valyi: On approximate vector optimization. Working paper 86-7, IIASA Laxenburg, 1986.

[27] P. Weidner: Ein Trennungskonzept und seine Anwendungen auf Vektoroptimierungsverfahren. Dissertation B, Matin-Luther-Universitt Halle- Wittemberg, 1991.

[28] T. Weir: A duality theorem for a multiple objective fractional optimization problem. Bull. Austral. Math. Soc. 34, 1986, 415-425.